DEPT OF MATHS FUNAAB 2019 MTS 101 TUTORIAL QUESTIONS5

WRITE YOUR SOLUTIONS NEATLY ON CLEAN WHITE PLAIN SHEETS

KEEP YOUR SOLUTIONS FOR FUTURE USE

REPORT ANY STUBBORN PROBLEM TO YOUR TUTORIAL TEACHERS

Roots of Quadratic Equations

1. If α and β are the roots of the equation $5x^2 - 6x - 7 = 0$, find the values of:

   (a) $(\alpha - \beta)^2$.

   (b) $\alpha^2 + \beta^2$.

   (c) $\alpha^3 + \beta^3$.

   (d) $\alpha^3 - \beta^3$.

   (e) $\alpha^4 + \beta^4$.

   (f) $\alpha^4 - \beta^4$.

   (g) $\alpha^6 + \beta^6$.

   (h) $\alpha^6 - \beta^6$.

2. The roots of the equation $9x^2 + 6x + 1 = 4kx$, where k is a real constant, are denoted by α and β.

   (a) Show that the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $x^2 + 6x + 9 = 4kx$.

   (b) Find the set of values of k for which α and β are real.

   (c) Find also the set of values of k for which α and β are real and positive.

3. (a) The roots of the equation $x^2 + px + q = 0$ are α and β.

      i. Given that the roots differ by $2\sqrt{3}$ and that the sum of the reciprocals of the roots is 4, find the possible values of p and q.

      ii. Find an equation whose roots of $\alpha + \frac{2}{\beta}$ and $\beta + \frac{2}{\alpha}$, expressing the coefficients in terms of p and q.

(b) Given that the roots of the equation $x^2 - x - 1 = 0$ are α and β, find, in its simplest form, the quadratic equation with numerical coefficients whose roots are $\frac{1+\alpha}{2-\alpha}$ and $\frac{1+\beta}{2-\beta}$.

(c) Given that $f(x) \equiv x^2 + (k+2)x + 2k$, show that the roots of the equation $f(x) = 0$ are real for all real values of k.

i. Find the roots of $f(x-k) = 0$.

ii. Find the value of k for which the equation $f(x-k) - 2x = 0$ has roots $x = 0$ and $x = 7$. For this value of k, find the minimum value of $f(x-k) - 2x$.

iii. Given that $2c = 2 - k$, show that the roots of the equation $f(x-k) + c^2 = 0$ are equal, and find the value of these equal roots when $k = 1$.

4. (a) i. Find the values of k for each of which the quadratic equations $x^2 + kx - 6k = 0$ and $x^2 - 2x - k = 0$ have a common root.

ii. If the quadratic equation $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, show that the solutions of $2x^2 + (a+b)x = (a+b)^2$ are $x = 1$ and $x = -\frac{1}{2}$.

iii. For what values of k does the equation $10x^2 + 4x + 1 = 2kx(2-x)$ have real roots ?

(b) Find the values of k for which the equation $x^2 + (k-3)x + k = 0$ has:

i. Real distinct roots.

ii. Roots of the same sign.

iii. Equal roots.

iv. Complex roots.

5. (a) If the difference between the roots of the equation $ax^2 - bx + c = 0$ is the same as the difference between the roots of the equation $bx^2 - cx + a = 0$, show that

$$b^4 - a^2c^2 = 4ab(bc - a^2).$$

Find the condition that the two equations should have a common root.

(b) The roots of the equation $2x^2 + x + 5 = 0$ are α and β, and those of the equation $2x^2 - 3x + 2k = 0$ are $\alpha + 1$ and $\beta + 1$. Find the value of k.

(c) Show that the expression

$$x^2 - (a + b + c)x + a^2 + b^2 + c^2 + 2bc - ca - ab$$

can never be negative if $x, a, b, c \in \mathbb{R}$.

(d) Obtain the roots of the equation

$$bcx^2 + [b^2 + c^2 - a(b + c)]x + (a - b)(a - c) = 0.$$

Find the quadratic equation whose roots are the reciprocals of the roots of this equation.

6. (a) If one root of the equation $ax^2 + bx + c = 0$ is double the other root, show that

   i. the roots are $-\frac{b}{3a}$ and $-\frac{2b}{3a}$ and

   ii. $2b^2 = 9ac$.

   (b) Find the value of k for which the equation $(x - 2)(x - 3) = k$ has roots which differ by 2.

   (c) If the roots of the equation $ax^2 + bx + c = 0$ differ by 1, show that they are $\frac{a-b}{2a}$ and $-\frac{a+b}{2a}$, and show that $b^2 = a(a + 4c)$.

   (d) Factorize $x^3 - 2x^2 + 4x - 3$. Hence, find the sum and product of the roots of the equation $x^3 - 2x^2 + 4x - 3 = 0$.

   (e) In the quadratic equation $x^2 - 2kx + 3k + 4 = 0$, k is a real number.

   i. If the equation has equal roots, find the values of k and solve the equation for each value of k.

   ii. If the equation has no real roots, find the range of values of k.

Graphs of Quadratic Functions, Maximum and Minimum Problems involving Quadratic Equations

1. Sketch the graphs of the following functions by clearly stating max/min values, intersection with axes, lines of symmetries, etc

   (a) $f(x) = 70x - \frac{5}{3}x^2 - 20$.

   (b) $f(x) = ab + (a - b)x - x^2$ where a and b are real constants.

   (c) $f(x) = 6x^2 - 13x + 6$.

   (d) $f(x) = 30 - 11x - 30x^2$.

   (e) $f(x) = x^4 - 20x^2 + 64$

   (f) $f(x) = ax^2 + bx - 2$ where $a < 0$ and b are real numbers.

   Find the numerical values of a and b given that $f(x)$ has a maximum at the point $(1, -5)$.

2. (a) If the line $y = x + k$, meets the curve $y = x^2 - x - 5$ at real points, find the minimum value of k.

   (b) The expression $ax^2 + bx + c$ is positive only when x lies between 1 and 3. Show that a is negative and find the range of values of x for which $ax^2 - bx + c$ is positive.

   (c) Given that a, b, c are real numbers, show that the roots of the equation $(x-a)(x-b) = c^2$ are real. State the condition for the roots to be equal.

   (d) Find the least value of the expression $x^2 - x$.

   For what values of x is the expression

$$x(x^2 - x - 2)(x^2 - x + 1)$$

   positive ?

   (e) Show that

$$\frac{5}{2x^2 + 3x + 3}$$

   is positive for all real values of x and find its greatest value.

   (f) If p, q are both real and $q > 4$, show that

$$\frac{x^2 + px + p}{x^2 + qx + q}$$

cannot be between $\frac{p}{q}$ and $\frac{p-4}{q-4}$ when x real.

3. (a) Find the set of possible values of k if

$$\frac{x^2+k}{x-1}$$

can take all real values when x is real. Find the set of possible values of $\frac{x^2+k}{x-1}$ when $k=3$.

(b) If $y = \frac{x^2+2x+\lambda}{2x-3}$ and x is real, find the greatest value of λ for which y can take all real values.

(c) Find the limits between which k must lie in order that

$$\frac{kx^2-6x+4}{4x^2-6x+k}$$

may be capable of all values when x is real.

(d) Show that for all real values of x, the expression

$$\frac{x+2}{x^2+3x+6}$$

cannot be greater than $\frac{1}{3}$, nor less than $-=fr15$. Find for what values of x, if any, it attains these values.

(e) Show that if x is real,

$$\frac{(x-1)(x-5)}{(x-2)(x-4)}$$

cannot lie between 1 and 4. Can it attain these two values and if so for what values of x ?

(f) Show that the value of the expression

$$\frac{(x+2)^2}{x+1}$$

cannot lie between 0 and 4 if x is real.

(g) Show that if a, b, c are real, the roots of the equation

$$(a^2+b^2)x^2+2(a^2+b^2+c^2)x+(b^2+c^2))=0$$

are also real.

4. Find all the roots of the following equations.

   (a) $30\left(x+\frac{1}{x}\right)^2+11\left(x+\frac{1}{X}\right)-30=0.$

   (b) $x^4-20x^2+64=0.$

   (c) $x^6-2x^3+1=0.$

   (d) $x(x+1)+\frac{12}{x(x+1)}=8.$

   (e) $x^4+2x^3-x^2+2x+1=0.$

   (f) $\sqrt{x^2-3x+6}-\sqrt{x^2-3x+3}=1.$

   (g) $\sqrt{\frac{x-1}{3x+2}}+2\sqrt{\frac{3x+2}{x-1}}=3.$

   (h) $x^2+\frac{9}{x^2}-4\left(x+\frac{3}{x}\right)-6=0.$

   (i) $\frac{x-1}{x+1}\frac{x+2}{x-2}=2+\frac{2}{x+4}.$

   (j) $\sqrt{x-9}-\sqrt{x-16}=\sqrt{x-24}.$

Inequalities

1. Solve the following inequalities and draw appropriate number lines to display your solutions.

   (a) $\frac{2x-1}{x+3}<\frac{2}{3}.$

   (b) $\frac{x-1}{x-2}>\frac{x-2}{x-3}.$

   (c) $\frac{2x^2+5x+7}{3x+5}\geq 2.$

   (d) $\frac{2x^2-3x-5}{x^2+2x+6}<\frac{1}{2}.$

   (e) $\mid x+3\mid=2.$

   (f) $\mid\frac{x-3}{x+1}\mid<2.$

   (g) $\frac{(2x-1)(x+4)}{x-5}>0.$

   (h) $\frac{1}{2}\leq\frac{x+3}{3}\leq\frac{3}{4}.$

   (i) $\mid\frac{x+1}{2}-\frac{x-1}{3}\mid\leq 1.$

   (j) $0\leq\log_{10}(2x-5)\leq 1.$

   (k) $\mid 2x^2+4x-11\mid\leq 5.$

(l) $2\left(\frac{2}{x} - \frac{1}{3}\right) > -\frac{1}{6}$.

(m) $\frac{1}{5x} > \frac{4}{15x} - \frac{1}{20}$.

(n) $\frac{5}{3x} + \frac{3}{4} \geq \frac{1}{12}$.

(o) $\frac{6}{x^2} < 6 + \frac{5}{x}$.

(p) $1 \leq \log_{10}(x^2 + x + 10) \leq 2$.

2. (a) If x and y are positive numbers, show that:

   i. $x + \frac{1}{x} \geq 2$ and

   ii. $(x + y)\left(\frac{1}{x} + \frac{1}{y}\right) \geq 4$.

   (b) If x, y, z are three positive numbers, show that

$$(x + y)(y + z)(z + x) \geq 8xyz.$$

   (c) If $x + y > 0$, show that

$$x^3 + y^3 > x^2y + xy^2.$$

   (d) Show that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

   Hence show that if x, y, z are all positive, then

$$x^3 + y^3 + z^3 > 3xyz.$$

   (e) i. The dimensions of a rectangle are $(x - 1)$cm and $(3x + 2)$cm. What is the range of values of x, if the area of the rectangle must not be less than $12cm^2$ or greater than $22cm^2$?

   ii. The two shorter sides of a right-angled triangle are $x$cm and $(x + 1)$cm. If the hypotenuse is to be longer than 5 cm but shorter than $\sqrt{85}$cm, find the range of possible values of x.

   iii. A rectangle of length $2x$cm is fitted inside a right-angled isosceles triangle. Show that the area of the rectangle is $2x(5 - x)cm^2$. Find the range of values of x if the area of the rectangle is to lie between $8cm^2$ and $12cm^2$.