DEPT OF MATHS FUNAAB 2019 MTS 101 TUTORIAL QUESTIONS3
WRITE YOUR SOLUTIONS NEATLY ON CLEAN WHITE PLAIN SHEETS
SUBMIT YOUR SOLUTIONS THROUGH YOUR CLASS REP ON MONDAY 08/07/19 AT EXACTLY 10 AM

Sets

1. (a) Let $P = \{3, 5, 7\}, Q = \{2, 4, 6\}, R = \{1, 9\}$ be subsets of $X = \{x \in \mathbb{N} : 1 \leq x \leq 10\}$. Compute the following:

   i. 2^P

   ii. $(P \triangle Q \triangle R)^c$

   iii. $P \times Q \times R$.

   (b) To investigate the popularity of three brands of soap X, Y, Z produced in a soap Industry, 150 Housewives were asked to fill Questionnaires and the following information was obtained: 60 Housewives had used X, 85 had used Y and 72 had used Z, 25 used X and Y, 35 had used X and Z, 35 had used X and Z, 17 had used Z and X.

   i. How many Housewives had used all the three brands ?

   ii. How many Housewives had used just two of the brands ?

   (c) Let A and B be nonempty subsets of the universal set X. Show that:

   i.
   $$[(A \cup A') \cap (B \cup B')] \cup (A \cap B) = X$$

   ii.
   $$(A \cup B) \cap (B \cup A') = B$$

   iii.
   $$(A \cup B)' = A' \cap B'$$

   iv.
   $$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

2. (a) Let A, B, C be nonempty subsets of the universal set X. Show that:

   i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   iii. $(A \cup B)' = A' \cap B'$

   iv. $(A \cap B)' = A' \cup B'$.

   (b) Using your results in (a) or otherwise, show that:

i. $(A' \cap B \cap C') \cup (A' \cap B \cap C) = B - A$

ii. $(A \cup B) \cap (B \cup A') = B$

iii. $(A \cap B) \cup (A \cap C) \cup (A \cap B' \cap C') = A$

iv. $A - (B \cap C) = (A - B) \cup (A - C)$.

(c) In a survey of 100 families, the number that read recent issues of a certain monthly magazine were found to be: April 26, June 48, April only 18, April but not May 23, April and June 8, June and May 8, none of the three months 24.

i. Draw a Venn diagram to represent the data and hence find how many families read:

ii. April, May and June issues

iii. May issue only

iv. May issue

v. June issue only

vi. June but not May issue.

vii. If a family is chosen at random, what is the probability that the family read issues in one month only ?

Binary Operations

3. The operation $*$ is defined over $\mathbb{R}$ the set of real numbers by

$$p * q = p + q - \frac{1}{2}pq.$$

(a) Show that $*$ is commutative and associative.

(b) Find the identity element for the operation $*$.

(c) Find the inverse (under $*$) of the real number p, stating any value of p for which no inverse exists.

(d) Determine whether or not

$$p * (q + r) = (p * q) + (p * r), \qquad \forall\ p, q, r \in \mathbb{R}.$$

4. Let X be a nonempty set with associative binary operation $\circ$. Let $x, y, z \in X$. Suppose x commutes with y and z, show that x commutes also with $y \circ z$.

5. Consider the set I of ordered pairs

$$I = \{(m, n) : \text{m, n are natural numbers}\}.$$

An operation $\oplus$ is defined on I by

$$(a, b) \oplus (c, d) = (a + c, b + d).$$

Show that this operation is commutative and associative.

Any two elements (a,b), (c,d) in I are to be considered equal if and only if $a + d = b + c$. Show that any element of the form (n,n) may be regarded as a neutral element with respect to $\oplus$.

Given that (r,s) is an inverse of (p,q), find the relationship between p, q, r, s. Hence find an inverse for the element (7,5) and an inverse for the element (m,n).

6. A binary operation $*$ is defined over the set $\mathbb{R}$ of real numbers by

$$x * y = x + y - x^2 y.$$

   (a) Determine whether or not $*$ is commutative and associative.

   (b) Evaluate:

      i. $2 * 3$.

      ii. $-5 * 4$.

      iii. $3 * (4 * 5)$.

   (c) Find the value(s) of x for which:

      i. $4 * x = 34$.

      ii. $(3 * x) + (x * 3) = 8$.

7. The function f is defined by

$$f(x) = 3x - 2, \qquad x \in \mathbb{R}.$$

   (a) The binary operation $\circ$ on the set $\mathbb{R}$ is such that

$$f(p \circ q) = f(p) \times f(q) \qquad \forall\, p, q \in \mathbb{R}.$$

      i. Show that $p \circ q = 3pq - 2p - 2q + 2$.

      ii. Show that $\circ$ is commutative and associative.

      iii. Find the identity element for the operation.

      iv. Find the inverse (under $\circ$) of the real number p, stating any value of p for which no inverse exists.

   (b) Another binary operation $\bullet$ on the set $\mathbb{R}$ is such that

$$f(p \bullet q) = \frac{f(p)}{f(q)}, \quad f(q) \neq 0 \quad \forall\, p, q \in \mathbb{R}.$$

i. Show that

$$p \bullet q = \frac{p + 2q - 2}{3q - 2}, \qquad q \neq \frac{2}{3}.$$

ii. Show that $\bullet$ is neither commutative nor associative.

iii. Determine whether or not

$$p \bullet (q \circ r) = (p \bullet q) \circ (p \bullet r) \quad \forall\, p, q, r \in \mathbb{R}.$$

8. Let S be the set of all ordered pairs $x = (x_1, x_2)$ with x_1 and x_2 real numbers. A binary operation $*$ is defined on S by

$$a * b = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)\,.$$

Show that this operation is commutative and associative.

Determine the identity element for this operation, and also the inverse of any element x. Hence solve:

$$a * x = b$$

where $a = (3, 4), b = (5, 6)$.

9. Let X be a nonempty set with associative binary operation $\circ$. If e and f are elements of X such that $x \circ e = x$ and $f \circ x = x$ for all x in X, show that $e = f$.
Furthermore, if $x \circ y = e = z \circ x$, show that $y = z$.

10. For any two subsets X and Y of a universal set $\mathcal{Z}$, the operation $\bullet$ is defined by

$$X \bullet Y = (X \cap Y') \cup (Y \cap X'),$$

where X', Y' denote the complements of X and Y respectively. Show that:

(a) the operation is commutative;

(b) the empty set $\emptyset$ is the identity element for $\bullet$;

(c) every element is its own inverse.

11. Find the identity element, if it exists, and the inverse of 5 when each of the following operations is defined on $\mathbb{R}$ the set or real numbers.

(a) $p * q = p + q$

(b) $p * q = pq$

(c) $p * q = p + q + pq$

(d) $p * q = pq + 2p + 2q$

(e) $p * q = \sqrt{pq}$

(f) $p * q = \frac{p}{q} + \frac{q}{p}$

(g) $p * q = \frac{p}{q} - p$.

12. Let X be a nonempty set and let 2^X be a power set of X. A binary operation $\circ$ is defined on 2^X by

$$A \circ B = A \triangle B \quad \forall\ A, B \in 2^X.$$

(a) Show that $\circ$ is commutative and associative.

(b) Determine whether or not:

i. $A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$ for all $A, B, C \in 2^X$.

ii. $A \circ (B \cap C) = (A \circ B) \cap (A \circ C)$ for all $A, B, C \in 2^X$.

(c) Determine the identity element for $\circ$ and the inverse of any element $A \in 2^X$.

Surds

13. (a) Show that

$$\frac{2+\sqrt{3}}{\sqrt{3}-1} - \frac{\sqrt{3}-1}{2(2+\sqrt{3})} = 5.$$

(b) If $x = 3 - \sqrt{3}$, show that

$$x^2 + \frac{36}{x^2} = 24.$$

(c) Show that

$$\frac{7}{5\sqrt{(1-\frac{1}{50})}} = \sqrt{2}.$$

(d) If $a = \sqrt{5}$ and $b = \sqrt{2}$, simplify

$$\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}.$$

(e) Find $\sqrt{14 + 6\sqrt{5}}$.

(f) If $x = \frac{1}{2}(1-\sqrt{5})$, express $4x^3 - 3x$ in its simplest surd form.

14. (a) Express $\frac{3\sqrt{5}-\sqrt{3}}{2\sqrt{5}+3\sqrt{3}}$ in the form $a + b\sqrt{15}$ where a and b are rational numbers.

(b) Simplify the following:

i. $\frac{3-2\sqrt{2}-\sqrt{5}}{3+\sqrt{2}+2\sqrt{3}}$.

ii. $\frac{1-\sqrt{2}+\sqrt[3]{3}}{1+2\sqrt{2}-3\sqrt[3]{3}}$.

(c) Rationalize the numerators of the following surds:

i. $\frac{3\sqrt{5}-2\sqrt{7}}{2\sqrt{5}-3\sqrt{7}}$.

ii. $\frac{3-3\sqrt{5}-2\sqrt{7}}{2-2\sqrt{5}-3\sqrt{7}}$.

iii. $\frac{3\sqrt{a}-2\sqrt{b}}{2b-3a}$.

iv. $\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{\sqrt{a}-\sqrt{b}+\sqrt{c}}$.

Indices And Logarithms

15. (a) Simplify

$$\frac{(x^4yz^{-3})^2 \times \sqrt{x^{-5}y^2z}}{(xz)^{7/2}}.$$

(b) If $x = \sqrt[3]{p+q} + \sqrt[3]{p-q}$ and $p^2 - q^2 = r^3$, show that

$$x^3 - 3rx - 2p = 0.$$

(c) Solve for x given that:

i. $2 \times 27^x - 5 \times 9^x + 3^{x+1} = 3^x$.

ii. $2 \times 3^{2x+3} - 7 \times 3^{x+1} - 68 = 0$.

iii. $5^{2x-3} \times 3^{2x+1} = 2^{3x-2}$.

iv. $5^{2x} - 5^{x+1} + 6 = 0$.

v. $3^{2x-1} - 28 \times 3^{x-2} + 1 = 0$.

vi. $\log_x^5 \times \log_x^3 = 15$.

vii. $\log_4^x \times \log_8^{x^4} = 32$.

viii. $1 + \log_2^{(x^2-4x-16)} = \log_2^{(x^2-3x+4)}$.

ix. $\log(x+9) = 1 + \log(x+1) - log(x-2)$.

16. Evaluate the following:

(a) $\log_{1/4}^{64}$.

(b) $\log_7^{98} - \log_7^{30} + \log_7^{15}$.

(c) $\log_x^{5/7} + 2\log_x^{7/6} - \log_x^{5/6}$.

(d) $\frac{\log\sqrt{27}+\log\sqrt{8}-\log\sqrt{125}}{\log 6-\log 5}$.

17. (a) If $\log_3^{(x-6)} = 2y$ and $\log_2^{(x-7)} = 3y$, show that

$$x^2 - 13x + 42 = 72^y.$$

Given that $y = 1$, find the possible value(s) of x.

(b) Given that $\log_8^{p-2)} + \log_8^q = r - \frac{1}{3}$ and $\log_2^{p-2)} - \log_2^q = 2r + 1$, show that

$$p^2 = 4 + 32^r.$$

If $r = 1$, find possible values of p and q.

(c) i. If $\log_2^a = \log_4^b$, express b in terms of a without logarithms.

ii. Given that $1 + \log_3^p = \log_{27}^q$, obtain a relation between p and q without involving logarithms.

iii. Find the relationship between x and y not involving logarithms, if $\log_9^x = 2 + \log_3^y$.

18. (a) If $m, n, x \in \mathbb{Z}^+$, show that

$$\log_{mn}^x = \frac{\log_n^x}{1 + \log_n^m}.$$

(b) By putting $x = \log a, y = \log b, z = \log c$ in the identity

$$x(y - z) + y(z - x) + z(x - y) = 0,$$

show that

$$\left(\frac{b}{c}\right)^{\log a} \times \left(\frac{c}{a}\right)^{\log b} \times \left(\frac{a}{b}\right)^{\log c} = 1$$

where logarithms are taken to any base.

(c) If $x^2 + y^2 = 7xy$, show that

$$\log(x + y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y.$$

(d) Show that:

i. $\log_c^a + \log_c^b = \log_c^{ab}$.

ii. $\log_a^b \times \log_b^c \times \log_c^a = 1$.

iii. $\frac{1}{\log_a^{abc}} + \frac{1}{\log_b^{abc}} + \frac{1}{\log_c^{abc}} = 1$.

(e) If $a = \ln(1 + 1/15)$, $b = \ln(1 + 1/24)$, $c = \ln(1 + 1/30)$, show that:

i. $ln2 = 7a + 5b + 3c$.

ii. $\ln 3 = 11a + 8b + 5c$.

iii. $\ln 5 = 16a + 12b + 7c$.

(f) Show that $x^{\log_x^y} = y$ for any positive real numbers x and y. Hence show that

$$81^{\frac{1}{\log_5^9}} + 3^{\frac{4}{\log_{\sqrt{6}}^3}} - \left(\sqrt{7}\right)^{\frac{2}{\log_{25}^7}} - 5^{\log_{25}^6} = 36 - \sqrt{6}.$$